Pricing credit derivatives under a correlated regime-switching hazard processes model

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Abstract In this paper, we study the valuation of a single-name credit default swap and a kth-to-default basket swap under a correlated regime-switching hazard processes model. We assume that the defaults of all the names are driven by a Markov chain describing the macro-economic conditions and some shock events modelled by a multivariate regime-switching shot noise process. Based on some expressions for the joint Laplace transform of the regime-switching shot noise processes, we give explicit formulas for the spread for a CDS contract and the kth-to-default basket swap.

Key words: hazard process, Markov chain, kth-to-default basket swap, multivariate regime-switching shot noise process

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1. Introduction

Portfolio credit derivatives have attracted considerable attention over the last decade among institutional investors. The difficulty in the valuation of such financial derivatives is the modeling of the default dependence among them. One of the most popular methods for modeling credit risk is the reduced-form approach, in which the time of default of a firm is defined as the first jump time of a point process.

There are two major types of reduced form models for describing default dependence, namely bottom-up models and top-down models. In the former approach, one focuses on modeling default intensities of individual reference entities and their aggregation to form a portfolio default intensity. Some works on bottom-up models include Duffie and Gârleanu (2001), Jarrow and Yu (2001) and Giesecke and Goldberg (2004). In the latter approach, one concerns modelling default at portfolio level. A default intensity for the whole portfolio is modelled without reference to the constituent names. Some procedures such as random thinning can be used to recover the default intensities of the individual entities. Some works on top-down models include Brigo et al. (2010), and Ding et al. (2009). We focus on bottom-up models.

There exist four major approaches to introduce default correlation within the reduced-form framework: the conditionally independent approach, the copula approach, the default contagion models, and the common shock models. In the conditionally independent default models, one may set the default intensities of the firms in the portfolio to be driven by a common set of macro-economic factors. Therefore, conditional on the realization of the macro-economic state variables, the default times are mutually independent; see, for example, Duffie and Gârleanu (2001) and Graziano and Rogers (2009). In the copula models, the dependence structure is linked through a copula function; see, for example, Schonbucher and Schubert (2001) and Hull and White (2004). Default contagion is another approach to model the default correlation. The contagion models study the direct interaction of firms in which the default probability of one firm may change upon defaults of some other firms in the portfolio; see, for example, Davis and Lo (2001) and Dong et al. (2016). The common shock models are based on the idea that a firm's default is driven by exogenous events, for example, policy events, natural catastrophes events, etc. Therefore, simultaneous defaults may occur under the common shock models;

see, for example, Giesecke (2003), Brigo et al. (2010), and Bielecki et al. (2012). This paper focuses on a conditionally independent approach.

The challenge in the valuation of portfolio credit derivatives using a conditionally independent approach is to describe the modelling of the default intensity or hazard process in the single firm case. Affine processes are usually used to model the default intensity, since they allow for explicit solutions to many important quantities in derivative pricing. See for example, Duffie et al. (2003). However, empirical studies point to the existence of different regimes in the default risk valuation, see, for example, Davies (2004) and Alexander and Kaeck (2008). Credit derivatives are long term instruments and thus it is very important to develop more appropriate models for valuation and risk management of credit products, which can take into account changes of market regimes or environments due to the crisis.

Markov regime-switching models have been widely used in different branches of modern financial economics to capture changes in market regimes. For example, see Buffington and Elliott (2002), Shen and Siu (2013), Elliott and Siu (2011) and others. Regime switches are often interpreted as structural changes in macro-economic conditions and in different stages of business cycles. Intuitively, default risk typically declines during economic expansion because strong earnings keep overall defaults rates low. Default risk increases during economic recession because earnings deteriorate, making it more difficult to repay loans or make bond payments. See for example, the U.S. subcrisis initially lead to the bankruptcy of several major subprime mortgage lenders and the decrease in the firm values and stock prices of some of the major lenders, including Countrywide Financial, Washington Mutual, Citigroup, and others. Then this crisis quickly became a global crisis and had significant impact on the values of credit derivatives not directly related to the mortgage financial institutions. Therefore, the subcrisis is the disaster to the whole people and it is reasonable to assume that most companies are more likely to default in the financial crisis. This motivates the quest for regime-switching models for valuation and risk management of credit products, which can incorporate the change in regimes of credit markets.

Recently, by an empirical analysis of the corporate bond market over the course of the last 150 years, Giesecke et al. (2011) point out that there exist three regimes, associated with high, middle, and low default risk. They also study the relationship between the default risk and the financial and macroeconomic variables by using the regime-switching model. However, they do not consider the modelling of the default dependence within the regime-switching framework. Motivated by them, we shall investigate a regime-switching model for correlated defaults and the pricing of the kth-to-default basket swap. Instead of studying the determinants of corporate default risk within a regime-switching framework in Giesecke et al. (2011), we directly model the hazard processes by some regime-switching processes, which incorporate both macroeconomic risks and firm-specific jump risks, so that we can derive some closed-form formulas for the portfolio credit derivatives.

Note that, within the reduced-form framework, Dong et al. (2014) consider a common shock model, in which the default intensities are modelled by a multivariate regime-switching process. Dong et al. (2016) investigate a default contagion model, where the default intensities are also described by some regime-switching processes. However, in Dong et al. (2014) and Dong et al. (2016), only the model parameters may switch whenever transitions in the Markov chain occur and therefore the hazard processes are absolutely continuous. Intuitively, if a Markov chain jumps from a good economic state to a bad economic state, this may cause the conditional default intensity to increase. This is because macro-economic risks are usually structural in nature. In addition to a jump component caused by exogenous factors, we consider a Markov regime-switching hazard process which can incorporate jumps due to structural changes in economic conditions into the dynamics of hazard process.

In this paper, we use a conditionally independent default approach different from those in Dong et al. (2014) and Dong et al. (2016), who adopt a common shock and a contagion model, respectively. The hazard processes we consider are modelled by some regime-switching processes, which are not absolutely continuous. The difference between our model and some existing regime-switching reduced-form models (such as, Dong et al. (2014) and Dong et al. (2016) is that only the model parameters can switch according to the Markov chain in the existing models, but in our hazard model, not only the model parameters may switch but also the hazard process may jump whenever transitions in the Markov chain occur. Therefore, in our proposed model, the jumps of the Markov chain may also trigger defaults, while defaults could not occur when the Markov chain transits from one state to another state in Dong et al. (2014) and Dong et al. (2016). Since the conditional default intensities of all the names in the portfolio go up when the economic state shifts from economic growth to recession in our model, we can increase the chances of observing a larger number of defaults in the portfolio during the economic depression. Therefore, the proposed hazard processes can well capture the clustering phenomena in correlated defaults.

This paper aims at providing a flexible and tractable model for correlated defaults which take into account the changes in market regimes due to financial crises. Under the proposed model, we can give analytic formulas for the CDS spreads. The rest of the paper is organized as follows. In section 2, we introduce a conditionally independent default model, in which the hazard processes are described by some dependent regime-switching pure jump processes. Section 3 derives the joint Laplace transform of the hazard processes. Based on the joint Laplace transform, we obtain the joint survival distributions. Section 4 gives the closed-form formulas for the CDS spread and the spread of the kth-to-default basket swap. Section 5 presents some numerical results. Finally, Section 6 concludes the paper.

2. The model

In this section, we model the dependent hazard processes within the reduced-form framework under a Markov environment. Consider a continuous-time model with a finite time horizon $\mathcal{T} = [0, T^*]$, where $T^* < \infty$. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered complete probability space, where P is the risk neutral measure and $\mathbb{F} := \{\mathcal{F}_t | t \in \mathcal{T}\}$ is a filtration satisfying the usual conditions of right continuity and completeness. Throughout the paper, it is assumed that all random variables are well defined on this probability space and $\mathcal{F}_{T^*}-$measurable.

Assume that there exists a homogeneous Markov chain $\mathbf{X} := \{X_t | t \in \mathcal{T}\}$ with generator $Q = (q_{ij})$, describing the macro-economic conditions. The state space of $\mathbf{X}$ can be taken to be, without loss of generality, the set of unit vectors $\{e_1, e_2, \cdots, e_N\}$, where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)^* \in R^N$ with the symbol $*$ denoting the transpose of a vector or a matrix. Let $\mathbb{H}^X := \{\mathcal{H}_t^X | t \in \mathcal{T}\}$ is a right-continuous, P-complete, natural filtration generated by the Markov chain $\mathbf{X}$. Elliott et al. (1994) provide the following semi-martingale decomposition for X_t:

$$dX_t = Q^* X_t dt + d\mathbf{M}_t, \tag{2.1}$$

where $\mathbf{M}_t$ is an $(\mathbb{H}^X, P)-$martingale.

Let $\langle ., . \rangle$ denote a scalar product in R^N, that is, for any $\mathbf{x}, \mathbf{y} \in R^N$, $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^N x_i y_i$. For each $k, j = 1, 2, \cdots, N$, and $j \neq k$, let $\mathbf{J}_{jk} := \{J_{jk}(t) | t \in \mathcal{T}\}$ denote the number of jumps by time t from state e_j to state e_k. That is,

$$\begin{aligned}
J_{jk}(t) &= \sum_{0<s\leq t} \langle X_{s-}, e_j \rangle \langle X_s, e_k \rangle \\
&= \int_0^t \langle X_{s-}, e_j \rangle \langle dX_s, e_k \rangle \\
&= \int_0^t \langle X_{s-}, e_j \rangle \langle Q^* X_s, e_k \rangle ds + \int_0^t \langle X_{s-}, e_j \rangle \langle d\mathbf{M}_s, e_k \rangle.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\overline{J}_{jk}(t) &:= J_{jk}(t) - \int_0^t \langle X_{s-}, e_j \rangle \langle Q^* X_s, e_k \rangle ds \\
&= J_{jk}(t) - q_{jk} \int_0^t \langle X_{s-}, e_j \rangle ds,
\end{aligned}$$

is an $(\mathbb{H}^X, P)-$martingale.

Assume that the discount factor is given by $D(0,t) = \exp\{-\int_0^t r_s ds\}$, where the interest rate r_t has the form

$$r_t = r(X_t), . \tag{2.2}$$

Here $r(X_t) = \langle \mathbf{r}, X_t \rangle$, where $\mathbf{r} = (r_1, r_2, \cdots, r_N)^* \in R^N$ with $r_i > 0$, for each $i = 1, 2, \cdots, N$.

Consider a portfolio of m credit-risky assets. For each $i = 1, 2, \cdots, m$, let τ_i denote the default time of the name i. Define

$$D_t^i = 1_{\{\tau_i \le t\}}, i = 1, 2, \cdots, m$$

and denote by the filtration $\mathbb{G} =: \{\mathcal{G}_t | t \in \mathcal{T}\}$ where $\mathcal{G}_t = \mathcal{G}_t^1 \vee \mathcal{G}_t^2 \vee \cdots \mathcal{G}_t^m$, with $\mathcal{G}_t^i = \sigma(D_s^i, s \le t)$. Assume that $\mathbb{F}$ is the enlarged filtration $\mathbb{H} \vee \mathbb{G}$ where $\mathbb{H} := \{\mathcal{H}_t | t \in \mathcal{T}\}$ is the filtration related to the underlying risk factors. Let us denote by

$$S_t^i = P(\tau_i > t | \mathcal{H}_t)$$

the survival process of τ_i with respect to a filtration $\mathbb{H}$. We postulate that $S_0^i = 1$ and $S_t^i > 0$ for every $t > 0$. Then for each $i = 1, 2, \cdots, m$, the $\mathbb{H}$-hazard process $\mathbf{\Lambda}^i := \{\Lambda^i(t) | t \in \mathcal{T}\}$, is defined by

$$\Lambda_t^i = -\log(S_t^i), t > 0.$$

Now we begin to describe the hazard processes for the default times. For each $i = 1, 2, \cdots, m$, the evolution of the hazard process $\mathbf{\Lambda}^i$ is governed by the following process:

$$\Lambda_t^i = \int_0^t \lambda_s^i ds + \sum_{j=1}^{N} \sum_{l=1, l \ne j}^{N} w_{jl}^i J_{jl}(t), \quad t \in \mathcal{T}, \tag{2.3}$$

where the constants $w_{jl}^i \ne 0$, for $j, l = 1, 2, \cdots, N, l \ne j$, and the process $\boldsymbol{\lambda}^i := \{\lambda_t^i | t \in \mathcal{T}\}$ is modelled by:

$$\lambda_t^i = \lambda^i(X_t) + \int_0^t e^{-a^i(t-s)} dJ_s^i \doteq \lambda^i(X_t) + L_t^i, \quad t \in \mathcal{T}. \tag{2.4}$$

Here, $\lambda^i(X_t) = \langle \boldsymbol{\lambda}^i, X_t \rangle$, where $\boldsymbol{\lambda}^i = (\lambda^{i1}, \lambda^{i2}, \cdots, \lambda^{iN})^* \in R^N$ with $\lambda^{ij} > 0$ for $i = 1, 2, \cdots, m$,$j = 1, \cdots, N$; $a^i > 0$ is a constant; and

$$J_t^i = \sum_{j=1}^{N_t^i + N_t^0} Y_j^i$$

is a regime-switching compound Poisson process, where N_t^l is a regime-switching Poisson process with intensity $\mu_l(s) = \langle \boldsymbol{\mu}_l, X_s \rangle$ for a positive vector $\boldsymbol{\mu}_l = (\mu_l^1, \cdots, \mu_l^N)^*$, for each $l = 0, 1, 2, \cdots, m$; Y_n^i is the size of the nth jump. Given $\Im_t^X$, it is assumed that $N_t^0, N_t^1 \cdots, N_t^m$ are mutually independent, and that $\{Y_j^1, j = 1, 2, \cdots\}, \cdots, \{Y_j^m, j = 1, 2, \cdots\}$ are mutually independent and independent of $N^0(t), \cdots, N^m(t)$. Furthermore, given the path of the Markov chain $\mathbf{X}$, we assume that for each $i = 1, 2, \cdots, m$, the jump sizes $Y_j^i,\ j = 1, 2, \cdots$ have a common conditional density f_t^i concentrated on $(0, \infty)$, where $f_t^i(.) = \langle \mathbf{f}^i(.), X_t \rangle$, for a vector $\mathbf{f}^i(.) = (f^{i1}(.), \cdots, f^{iN}(.))^*$.

Remark 2.1 *Although λ_t^i in (2.3) jumps upward for each $i = 1, 2, \cdots, m$, Λ_t^i can take negative values with positive probability since the constants $w_{jl}^i, j, l = 1, \cdots, N, j \ne l, i = 1, \cdots, m$ are not necessarily non-negative. However, in practical applications, the absolute value of $\sum_{j=1}^{N} \sum_{l=1, l \ne j}^{N} w_{jl}^i J_{jl}(t)$ is usually much smaller than the value of $\int_0^t \lambda_s^i ds$, so that the probability Λ_t^i takes negative values can be considered negligible. This is because the values of $w_{jl}^i, j, l = 1, \cdots, N, j \ne l, i = 1, \cdots, m$ are small,* *and a lot of empirical results show that the regime with low-default rate is usually very persistent. See for example, the empirical results presented in Giesecke et al. (2011) show that being in the regime with mid-default rate is an event that occurs roughly every decade, on average, while being in the regime with high-default rate occurs about every fifty years, on average.*

Note that, for each $i = 1, \cdots, m$, $\mathbf{L}^i = \{L_t^i | \in \mathcal{T}\}$ is a mean-reverting regime-switching Markov process, and it solves the stochastic differential equation

$$dL_t^i = -a^i L_t^i dt + dJ_t^i, \quad L_0^i = 0. \tag{2.5}$$

If there is no regime switching, then L_t^i follows a shot noise process. In the literature, the shot noise processes are good tools for describing the arrival intensities as they allow for explicit solutions to many important quantities in derivative pricing. For example, Gaspar and Schmidt (2010) consider a multivariate default model driven by the shot noise processes and show that the shot noise processes can describe historical data very well and give a better fit in calibration than the affine jump-diffusion models proposed by Duffie and Gârleanu (2001). As is pointed out by Dassios and Jang (2003), the shot noise process measures the frequency, magnitude and time period needed to go back to the previous level of intensity immediately after shock events occur. This paper extends the shot noise process to a regime-switching version. Intuitively, the hazard process of a firm depends on a firm specific term and some common factors. The processes $\lambda_t^1, \cdots, \lambda_t^m$ given by (2.4) which are driven by a multivariate regime-switching shot noise process with common jumps capture the impacts of idiosyncratic and common factors on the hazard processes.

We can see that the hazard processes jump whenever a transition of the Markov chain occurs. Intuitively, if the chain jumps from a state of economic growth to a state of economic recession, this systematic risk may cause the conditional default intensity of all the firms to go up. Therefore, the default dependence modelled by (2.3)-(2.4) stems from the Markov chain $\mathbf{X}$ and the common jumps in $L_t^i, i = 1, 2, \cdots, m$.

Let $\mathbb{H}^{L^i} := \{\mathcal{H}_t^{L^i} | t \in \mathcal{T}\}$ be the right continuous and $P-$complete, natural filtration generated by $\mathbf{L}^i$, for each $i = 1, 2, \cdots, m$. Then for each $t \in \mathcal{T}$, $\mathcal{H}_t$ is the enlarged $\sigma-$field $\mathcal{H}_t^X \vee \mathcal{H}_t^{L^1} \vee \cdots \vee \mathcal{H}_t^{L^m}$.

Assume the default times $\tau_1, \tau_2, \cdots, \tau_m$ are mutually conditionally independent given $\mathbb{H}$, that is, for any $0 < t_i \leq T, i = 1, 2, \cdots, m$,

$$\begin{aligned} P(\tau_1 > t_1, \tau_2 > t_2, \cdots, \tau_m > t_m | \mathcal{H}_T) &= \prod_{i=1}^m P(\tau_i > t_i | \mathcal{H}_T) \\ &= \prod_{i=1}^m e^{-\Lambda_{t_i}^i}. \end{aligned} \tag{2.6}$$

Therefore, in order to derive the joint survival probability, we shall give the conditional joint Laplace transform of $\Lambda_t^i, i = 1, \cdots, m$.

3. Laplace transforms and survival distributions

In this section, we shall derive explicit formulas for the marginal and joint survival probabilities.

For $c^i \geq 0, i = 0, 1, \cdots, m$, let

$$\begin{aligned} V(t,T) &= E\left[e^{-c^0 \int_t^T r_s ds - \sum\limits_{i=1}^m c^i (\Lambda_T^i - \Lambda_t^i)} X_T | \mathcal{H}_t \right], \\ &= E\left[e^{-\int_t^T (c^0 r(X_s) + \sum\limits_{i=1}^m c^i (\lambda^i(X_s) + L_s^i)) ds - \sum\limits_{j=1}^N \sum\limits_{k=1, k\neq j}^N \sum\limits_{i=1}^m c^i w_{jk}^i (J_{jk}(T) - J_{jk}(t))} X_T | \mathcal{H}_t \right]. \end{aligned}$$

Since $(X_t, L_t^1 \cdots, L_t^m)^*$ is an $(m+1)-$dimensional Markov process with respect to $\mathcal{H}_t$, we have

$$
\begin{aligned}
V(t,T) &= E\left[e^{-c^0 \int_t^T r_s ds - \sum_{i=1}^m c^i(\Lambda_T^i - \Lambda_t^i)} X_T | L_t^i, i = 1, \cdots, m, X_t\right] \\
&=: \theta(t, T, L_t^1, \cdots, L_t^m, X_t).
\end{aligned}
$$

Write

$$\theta_i = \theta(t, T, L_t^1, \cdots, L_t^m, e_i), i = 1, 2, \cdots, N,$$

and write

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_N)^* \in R^N.$$

The following result gives the explicit expression for $\theta(t, T, L_t^1, \cdots, L_t^m, X_t)$.

To make the following derivations mathematically rigorous, we shall impose the following integrability conditions:

1. $E\left[|e^{-c^0 \int_0^T r_s ds - \sum_{i=1}^m c^i \Lambda_T^i} X_T|\right] < \infty$;

2. $E[\theta(t, T, l^1, \cdots, l^i + Y^i, \cdots, l^m, x) - \theta_t] < \infty$ for each $i = 1, 2, \cdots, m$;

3.$E[\theta(t, T, l^1 + Y^1, \cdots, l^i + Y^i, \cdots, l^m + Y^m, x) - \theta_t] < \infty$;

4. $E\left[\int_0^T e^{-c^0 \int_0^t r_s ds - \sum_{i=1}^m c^i(\Lambda_T^i - \Lambda_t^i)} ||\theta_t||^2 dt\right] < \infty$ with the norm $||.||$ on R^N defined by

$$||\zeta|| = \zeta^*[\mathbf{diag}(QX_t) - \mathbf{diag}(X_t)Q^* - Q\mathbf{diag}(X_t)]\zeta,$$

where $\mathbf{diag}(\chi)$ is a diagonal matrix with diagonal entries given by the vector χ.

Theorem 3.1. *For $c^i \geq 0, i = 0, 1, \cdots, m$, we have*

$$V(t,T) = e^{-\sum_{i=1}^m B^i(t,T) L_t^i} \langle \Psi_1(t,T), X_t \rangle, \tag{3.1}$$

where $B^i(t,T) = -c^i(1 - e^{-a^i(T-t)})/a^i$, for $i = 1, \cdots, m$, and $\Psi_1(t,T)$ is the fundamental matrix solution of

$$\frac{d\Psi_1(t,T)}{dt} + \tilde{Q}(t)\Psi_1(t,T) = 0, \Psi_1(T,T) = \mathbf{I}.$$

Here $\mathbf{I}$ is an $N \times N$ identity matrix and $\tilde{Q}(t) = (\tilde{q}_{ij}(t))$ is an $N \times N$ matrix:

$$
\begin{aligned}
\tilde{q}_{ij}(t) &= q_{ij} e^{-\sum_{l=1}^m c^l w_{ij}^l}, i \neq j, \\
&= q_{ii} + G_t^i - c^0 r_i - \sum_{l=1}^m c^l \lambda^{li}, i = j,
\end{aligned}
$$

with

$$G_t^i = \sum_{l=1}^m \mu_l^i \left(\int_0^\infty e^{B^l(t,T)x} f^{li}(x) dx - 1\right) + \mu_0^i \left(\prod_{l=1}^m \int_0^\infty e^{B^l(t,T)x} f^{li}(x) dx - 1\right).$$

Furthermore,

$$E\left[e^{-c^0 \int_t^T r_s ds - \sum_{i=1}^m c^i(\Lambda_T^i - \Lambda_t^i)} | \mathcal{H}_t\right] = e^{-\sum_{i=1}^m B^i(t,T) L_t^i} \langle \Psi_1(t,T)\mathbf{1}, X_t \rangle, , \tag{3.2}$$

where $\mathbf{1} = (1, 1, \cdots, 1)^* \in R^N$.

Proof. We use the martingale approach to derive (3.1). For notational convenience, denote by $\theta_t = \theta(t, T, L_t^1, \cdots, L_t^m, X_t)$. Consider the function

$$\overline{V}(t,T) = U_t \theta_t,$$

where

$$U_t = \exp(-\int_0^t c^0 r(X_s) ds - \sum_{i=1}^m c^i \Lambda_t^i).$$

Applying Itô's differentiation rule to $\overline{V}(t,T)$ yields

$$\begin{aligned}
d\overline{V}(t,T) &= U_{t^-} d\theta_t + \theta_{t^-} dU_t + d[U,\theta]_t \\
&= U_t\Big(\frac{\partial}{\partial t} - \sum_{i=1}^m a^i L_t^i \frac{\partial}{\partial l^i}\Big)\theta_t dt + U_{t^-} \sum_{i=1}^m \Big(\big(\theta_t - \theta(t, T, L_t^1, \cdots, L_{t^-}^i, \cdots, L_t^m, X_t)\big) dN_t^i \\
&+ U_{t^-} \Big((\theta_t - \theta(t, T, L_{t^-}^1, \cdots, L_{t^-}^i, \cdots, L_{t^-}^m, X_t)\Big) dN_t^0 \\
&+ U_{t^-} \langle \boldsymbol{\theta}, Q^* X_t \rangle dt + U_{t^-} \langle \boldsymbol{\theta}, d\mathbf{M}_t \rangle - \Big(c^0 r(X_s) + \sum_{i=1}^m c^i (\lambda^i(X_s) + L_s^i)\Big) U_{t^-} \theta_t dt \\
&+ \theta_{t^-} U_{t^-} \sum_{j \neq k} 1_{\{X_{t^-} = e_j, X_t = e_k\}} (e^{-\sum\limits_{i=1}^m c^i w_{jk}^i} - 1) \\
&+ U_{t^-} \sum_{j \neq k} 1_{\{X_{t^-} = e_j, X_t = e_k\}} (e^{-\sum\limits_{i=1}^m c^i w_{jk}^i} - 1)(\theta_t - \theta_{t^-}),
\end{aligned}$$

where $\{[U,\theta]_t | t \in \mathcal{T}\}$ is the optional covariation of the processes $\{U_t | t \in \mathcal{T}\}$ and $\{\theta_t | t \in \mathcal{T}\}$.

As is pointed out in Remark 2.1, in practical applications, for each $i = 1, 2, \cdots, m$, the probability Λ_t^i takes negative values can be considered negligible since the absolute value of $\sum\limits_{j=1}^N \sum\limits_{l=1, l\neq j}^N w_{jl}^i J_{jl}(t)$ is usually much smaller than the value of $\int_0^t \lambda_s^i ds$. Therefore, $|\overline{V}(t,T)| \leq 1$ and then $\overline{V}(t,T)$ is a bounded $(\mathbb{H}, P)-$martingale. Furthermore, from the above integrability conditions, we can conclude that the bounded variation terms, which are not martingales, must sum to zero:

$$\begin{aligned}
&\Big(\frac{\partial}{\partial t} - \sum_{i=1}^m a^i l^i \frac{\partial}{\partial l^i} - (\langle \sum_{i=1}^m c^i \boldsymbol{\lambda}^i + c^0 \mathbf{r}, x \rangle + \sum_{i=1}^m c^i l^i)\Big) \theta(t, T, l^1, \cdots, l^m, x) \\
&+ \sum_{i=1}^m \langle \boldsymbol{\mu}^i, x \rangle E[\theta(t, T, l^1, \cdots, l^i + Y^i, \cdots, l^m, x) - \theta_t] \\
&+ \langle \boldsymbol{\mu}^0, x \rangle E[\theta(t, T, l^1 + Y^1, \cdots, l^i + Y^i, \cdots, l^m + Y^m, x) - \theta_t] \\
&+ \langle \boldsymbol{\theta}, Q^* x \rangle + \sum_{j=1}^N 1_{\{X_t = e_j\}} \sum_{j \neq k} q_{jk} (e^{-\sum\limits_{i=1}^m c_i w_{jk}^i} - 1) \theta(t, T, l^1, \cdots, l^m, k) = 0.
\end{aligned} \tag{3.3}$$

Due to the affine structure of L_t^i, motivated by Duffie et al. (2003), we try the solution

$$\theta(t, T, l^1, \cdots, l^m, x) = e^{\sum\limits_{i=1}^m B_i(t,T) l^i} C(t, T, x), \tag{3.4}$$

where the terminal conditions are given by

$$B_i(T,T) = 0, C(T,T,x) = x.$$

Write

$$\mathbf{C}(t,T) = (C(t,T,e_1),\cdots,C(t,T,e_N))^*.$$

Substituting the solution for θ given by (3.4) into (3.3) gives

$$\frac{\partial B_i}{\partial t} - a^i B_i(t,T) - c^i = 0, B_i(T,T) = 0, \tag{3.5}$$

and

$$\begin{aligned} &\frac{\partial C(t,T,e_j)}{\partial t} + C(t,T,e_j)(q_{jj} - c_0 r_j - \sum_{i=1}^m c^i \lambda^{ij} + \mu_0^j (\prod_{i=1}^m \int_0^\infty e^{B_i(t,T)y_i} f^{ij}(y_i) dy_i - 1) \\ + &\sum_{i=1}^m \int_0^\infty \mu_i^j (e^{B_i(t,T)y_i} - 1) f^{ij}(y_i) dy_i) + \sum_{k=1,k\neq j}^N q_{jk} (e^{-\sum_{i=1}^m c^i w_{jk}^i} - 1) C(t,T,e_k) = 0, \end{aligned} \tag{3.6}$$

with

$$C(T,T,e_j) = e_j, j = 1,2,\cdots,N.$$

From (3.5), it is easy to obtain that

$$B(t,T) = -c^i(1 - e^{-a^i(T-t)})/a^i.$$

It remains to prove that there exists a unique solution of (3.6). We can rewrite (3.6) as

$$\frac{\partial \mathbf{C}(t,T)}{\partial t} + \tilde{Q}(t)\mathbf{C}(t,T) = 0, \mathbf{C}(T,T) = \mathbf{I}.$$

Let $\Psi(t,T)$ denote the fundamental matrix solution of

$$\frac{d\Psi(t,T)}{dt} + \tilde{Q}(t)\Psi(t,T) = 0, \Psi(T,T) = \mathbf{I}. \tag{3.7}$$

Since $\tilde{Q}(t)$ is continuous, there exists a unique solution of (3.7) over the finite time interval $\mathcal{T}$. Hence,

$$C(t,T,x) = \langle \Psi(t,T), X_t \rangle.$$

Eq. (3.2) holds since

$$\begin{aligned} &E\left[e^{-\int_t^T c^0 r(X_s)ds - \sum_{i=1}^m c^i(\Lambda_T^i - \Lambda_t^i)} |\mathcal{H}_t\right] \\ = \; &E\left[e^{-\int_t^T c^0 r(X_s)ds - \sum_{i=1}^m c^i(\Lambda_T^i - \Lambda_t^i)} \langle X_T, \mathbf{1}\rangle |\mathcal{H}_t\right]. \end{aligned}$$

□

Remark 3.1 *If we let $c^0 = 1$ and $c^i = 0$ for each $i = 1,\cdots,m$ in Theorem 3.1, then we have*

$$E[e^{-\int_t^T r_s ds} X_T | X_t] = \langle e^{(Q - \mathbf{diag}(\mathbf{r}))(T-t)}, X_t \rangle, \tag{3.8}$$

and

$$E[e^{-\int_t^T r_s ds} | X_t] = \langle e^{(Q - \mathbf{diag}(\mathbf{r}))(T-t)} \mathbf{1}, X_t \rangle, \tag{3.9}$$

where $\mathbf{diag}(\bar{\mathbf{r}})$ *is a diagonal matrix with diagonal entries given by the vector* $\mathbf{r}$. *(3.8) and (3.9) are the same as the results given by LemmaA1 in Buffington and Elliott (2002).*$\square$

The following two results are direct consequences of Theorem 3.1.

Corollary 3.1. *For each* $l = 1, 2, \cdots, m$ *and any* $0 < t \le T$, *the survival distribution for* τ_l *is given by*

$$P(\tau_l > t) = \langle \Psi_2^l(0,t)\mathbf{1}, X_0\rangle, \tag{3.10}$$

where $\Psi_2^l(s,t)$ *is the fundamental matrix solution of*

$$\frac{d\Psi_2^l(s,t)}{ds} + \overline{Q}^l(s)\Psi_2^l(s,t) = 0, \Psi_2^l(t,t) = \mathbf{I}.$$

Here $\overline{Q}^l(s)$ *is an* $N \times N$ *matrix:*

$$\begin{aligned} \overline{q}_{ij}^l(s) &= q_{ij}e^{-w_{ij}^l}, i \neq j, \\ &= q_{ii} + \overline{G}_s^{li} - \lambda^{li}, i = j, \end{aligned}$$

where

$$\overline{G}_s^{li} = (\mu_l^i + \mu_0^i)(\int_0^\infty e^{\overline{B}^l(s,t)x} f^{li}(x)dx - 1), \tag{3.11}$$

with $\overline{B}^l(s,t) = -\frac{1-e^{-a^l(t-s)}}{a^l}$.

$\square$

Denote by $\mathcal{S}$ the collection of all the nonempty subsets of the set $\{1, \cdots, n\}$. The following result gives explicit formulas for the joint survival distributions of the default times.

Corollary 3.2. *For each* $s \in \mathcal{S}$ *and any* $0 < t \le T$, *we have*

$$P(\bigcap_{j\in s}\{\tau_j > t\}) = \langle \Psi_3^s(0,t)\mathbf{1}, X_0\rangle, \tag{3.12}$$

where $\Psi_3^s(v,t)$ *is the fundamental matrix solution of*

$$\frac{d\Psi_3^s(v,t)}{dv} + \overline{Q}^s(v)\Psi_3^s(v,t) = 0, \Psi_3^s(t,t) = \mathbf{I}.$$

Here $\overline{Q}^s(v)$ *is an* $N \times N$ *matrix:*

$$\begin{aligned} \overline{q}_{ij}^s(v) &= q_{ij}e^{-\sum_{l\in s} w_{ij}^l}, i \neq j, \\ &= q_{ii} + \tilde{G}_v^{si} - \sum_{l\in s}\lambda^{li}, i = j, \end{aligned}$$

where

$$\tilde{G}_v^{si} = \sum_{k\in s}\mu_k^i(\int_0^\infty e^{\overline{B}^k(v,t)x} f^{ki}(x)dx - 1) + \mu_0^i(\prod_{k\in s}\int_0^\infty e^{\overline{B}^k(v,t)x} f^{ki}(x)dx - 1), \tag{3.13}$$

with $\overline{B}^k(v,t)$ *given in Corollary 3.1.*

In particular, for any $0 < t \le T$, *the joint survival distribution for* $\tau_1, \cdots, \tau_m$ *is given by*

$$P(\tau_1 > t, \tau_2 > t, \cdots, \tau_m > t) = \langle \Psi_4(0,t)\mathbf{1}, X_0\rangle, \tag{3.14}$$

where $\Psi_4(s,t)$ *is the fundamental matrix solution of*

$$\frac{d\Psi_4(v,t)}{dv} + \hat{Q}(v)\Psi_4(v,t) = 0, \Psi_4(t,t) = \mathbf{I}.$$

Here $\hat{Q}(v)$ *is an* $N \times N$ *matrix:*

$$\begin{aligned} \hat{q}_{ij}(v) &= q_{ij}e^{-\sum\limits_{l=1}^{m} w_{ij}^l}, i \neq j, \\ &= q_{ii} + \hat{G}_v^i - \sum_{l=1}^{m} \lambda^{li}, i = j, \end{aligned}$$

with

$$\hat{G}_v^i = \sum_{k=1}^{m} \mu_k^i (\int_0^\infty e^{\overline{B}^k(v,t)x} f^{ki}(x)dx - 1) + \mu_0^i (\prod_{k=1}^{m} \int_0^\infty e^{\overline{B}^k(v,t)x} f^{ki}(x)dx - 1). \quad (3.15)$$

□

4. CDS and kth-to-default basket swap

In this section, we will investigate the pricing of the fair spreads of a single-name credit default swap and a kth-to-default basket swap.

4.1. Single-name credit default swap

In this subsection, we shall give the formula for the spread of a single CDS contract under our proposed regime-switching hazard process model.

A credit default swap (CDS) is a financial swap agreement between the buyer of the default protection on a reference risky entity and the seller of the default protection. The protection seller receives fixed periodic payments (CDS premium) from the protection buyer, in return for compensating the buyer's losses on the reference entity when a credit event occurs. This subsection focuses on valuing a single-name CDS contract on name i. Assume the default time of name i has the hazard process Λ_t^i modelled by (2.3). Consider a unit notional with the maturity T and a constant recovery R. The protection buyer pays an annualized premium to the protection seller before maturity T at specified time points $0 = t_0 < t_1 < \cdots < t_M \leq T$, with $\triangle t_i = t_i - t_{i-1}$. As soon as name i has defaulted, the protection buyer stops further premium payments and the protection seller is committed to paying the insurance buyer the default compensation $1 - R$, where R is the constant recovery rate.

We first describe the cash flows of a CDS on name i. The cash flows of a CDS are as follows:

Default leg: the protection seller covers the credit losses $1 - R$ as soon as name i has defaulted;

Premium leg: the protection buyer pays $s_i\Delta t_k$ to the seller, at each date $t_k, k = 1, \cdots, M$ until maturity or until name i defaults before maturity.

Then, the CDS spread on name i, s_i, is determined so that the discounted payoff of the two legs are equal when the contract is settled at the initial time. More precisely, the CDS spread s_i should satisfy

$$s_i E[\sum_{l=1}^{M} e^{-\int_0^{t_l} r_u du} \triangle t_l 1_{\{\tau_i > t_l\}}] = (1 - R)E[\sum_{l=1}^{M} e^{-\int_0^{t_l} r_u du} 1_{\{t_{l-1} < \tau_i \leq t_l\}}].$$

Hence,

$$s_i = \frac{(1-R)E[\sum_{l=1}^{M} e^{-\int_0^{t_l} r_u du} 1_{\{t_{l-1}<\tau_i \le t_l\}}]}{E[\sum_{l=1}^{M} e^{-\int_0^{t_l} r_u du} \triangle t_l 1_{\{\tau_i > t_l\}}]}. \tag{4.1}$$

Proposition 4.1 *The spread of the single-name CDS on name i, s_i, is given by*

$$s_i = \frac{\sum_{l=1}^{M} (\hat{P}^i_{1l} - \hat{P}^i_{2l})}{\sum_{l=1}^{M} \triangle t_l \hat{P}^i_{2l}}, \tag{4.2}$$

where

$$\hat{P}^i_{1l} = \langle \hat{\Psi}^i(0, t_{l-1})(e^{(Q-\mathbf{diag(r)})\Delta t_l}\mathbf{1}), X_0 \rangle,$$

and

$$\hat{P}^i_{2l} = \langle \hat{\Psi}^i(0, t_l)\mathbf{1}, X_0 \rangle.$$

Here $\hat{\Psi}^i(u,t)$ is the fundamental matrix solution of

$$\frac{d\hat{\Psi}^i(u,t)}{du} + O^i(u)\hat{\Psi}^i(u,t) = 0, \hat{\Psi}^i(t,t) = \mathbf{I}. \tag{4.3}$$

where $O^i(u) = (o^i_{kj}(u))$ is an $N \times N$ matrix:

$$\begin{aligned} o^i_{kj}(u) &= q_{kl} e^{-w^i_{kj}}, k \neq j, \\ &= q_{kk} + \overline{G}^{ik}_u - r_k - \lambda^{ik}, k = j, \end{aligned}$$

with $\overline{G}^{ik}_u$ defined in (3.11).

Proof. We first calculate the discounted payoff. Using the "tower property" of conditional expectation, we have

$$\begin{aligned} E[e^{-\int_0^{t_l} r_u du} 1_{\{t_{l-1}<\tau_i \le t_l\}}] &= E[e^{-\int_0^{t_l} r_u du} P(t_{l-1} < \tau_i \le t_l | \mathcal{H}_{t_l})] \\ &\hat{=} (\hat{P}^i_{1l} - \hat{P}^i_{2l}), \end{aligned}$$

where

$$\hat{P}^i_{1l} = E[e^{-\int_0^{t_l} r_u du - \Lambda^i_{t_{l-1}}}]$$

and

$$\hat{P}^i_{2l} = E[e^{-\int_0^{t_l} r_u du - \Lambda^i_{t_l}}].$$

Therefore, it remains to calculate $\hat{P}^i_{1l}$ and $\hat{P}^i_{2l}$. Again using the "tower property" of conditional expectation yields

$$\begin{aligned} \hat{P}^i_{1l} &= E\big[e^{-\int_0^{t_{l-1}} r_u du - \Lambda^i_{t_{l-1}}} E[e^{-\int_{t_{l-1}}^{t_l} r_v dv} | \mathcal{H}^X_{t_l} \vee \mathcal{H}_{t_{l-1}}]\big] \\ &= \langle e^{(Q-\mathbf{diag(r)})\Delta t_l}\mathbf{1}, E[e^{-\int_0^{t_{l-1}} r_u du - \Lambda^i_{t_{l-1}}} X_{t_{l-1}}] \rangle \\ &= \langle \hat{\Psi}^i(0, t_{l-1})(e^{(Q-\mathbf{diag(r)})\Delta t_l}\mathbf{1}), X_0 \rangle, \end{aligned}$$

where the second equality follows from (3.9), and the third equality is a direct consequence of Theorem 3.1.

The expression for $\hat{P}^i_{2l}$ can be directly obtained from Theorem 3.1,

$$\hat{P}^i_{2l} = \langle \hat{\Psi}^i(0, t_l)\mathbf{1}, X_0\rangle.$$

The discounted payoff of the premium leg is

$$E[e^{-\int_0^{t_l} r_v du} 1_{\{\tau_k > t_l\}}] = E[e^{-\int_0^{t_l} r_v dv} P(\tau_i > t_l | \mathcal{H}_{t_l})] = \hat{P}^i_{2l}.$$

Then substituting the expressions for $\hat{P}^i_{1l}$ and $\hat{P}^i_{2l}$ into the discounted payoffs of the premium leg and the default leg, we end the proof. $\square$

4.2. kth-to-default basket swap

A kth-to-default basket swap, which is a commonly traded product of portfolio credit derivatives, is a bilateral contract between an insurance buyer and an insurance seller. The payment streams of this derivative depend on the default times of an underlying portfolio of m credit-risky assets. Assume that the default dependence structure of the m underlyings is modelled by (2.3). Consider a unit notional with the maturity T and a constant recovery R. The protection buyer pays an annualized premium to the protection seller before maturity T at specified time points $0 = t_0 < t_1 < \cdots < t_M \leq T$, with $\triangle t_i = t_i - t_{i-1}$. As soon as k assets of the underlying portfolio have defaulted, the insurance buyer stops further premium payments and the protection seller is committed to paying the insurance buyer the default compensation $1 - R$, where R is the constant recovery rate.

Now we will introduce the definition of the kth-to-default time. Denote by

$$Y_t = \sum_{i=1}^{m} 1_{\{\tau_i > t\}}$$

be the number of names which still be alive at time t. Define the kth-to-default time as

$$\tau^k = \inf\{t : m - Y_t \geq k\} = \inf\{t : Y_t \leq m - k\}.$$

In particular, the stopping time

$$\tau^1 = \{t : Y_t \leq m - 1\} = \min_i\{\tau_i\}$$

is called the first-to-default time.

Then the conditional and unconditional distributions of τ^k are given by

$$\begin{aligned} P(\tau^k \leq t | \mathcal{H}_t) &= P(Y_t \leq m - k | \mathcal{H}_t) \\ &= \sum_{i=0}^{m-k} P(Y_t = i | \mathcal{H}_t), \end{aligned}$$

and

$$P(\tau^k \leq t) = P(Y_t \leq m - k) = \sum_{i=0}^{m-k} P(Y_t = i).$$

It is obvious that the distribution of τ^k depends on the distribution of Y_t. From Giesecke (2003), we have

$$\begin{aligned} P(Y_t = i | \mathcal{H}_t) &= \sum_{h=i}^{m} \left(\begin{array}{c} h \\ i \end{array}\right) (-1)^{h-i} \sum_{s \in \mathcal{S}, |s| = h} P(\bigcap_{j \in s}\{\tau_j > t\} | \mathcal{H}_t) \\ &= \sum_{h=i}^{m} \left(\begin{array}{c} h \\ i \end{array}\right) (-1)^{h-i} \sum_{s \in \mathcal{S}, |s| = h} e^{-\sum\limits_{j \in s} \Lambda_t^j}, \end{aligned}$$

where the last equality follows from the conditional independence of the default times.

Furthermore, the distribution of Y_t is given by

$$\begin{aligned} P(Y_t = i) &= \sum_{h=i}^{m} \binom{h}{i} (-1)^{m-i} \sum_{s\in\mathcal{S},|s|=h} P(\bigcap_{j\in s}\{\tau_j > t\}) \\ &= \sum_{h=i}^{m} \binom{h}{i} (-1)^{h-i} \sum_{s\in\mathcal{S},|s|=h} \langle \Psi_3^s(0,t)\mathbf{1}, X_0\rangle, \end{aligned}$$

where $\Psi_3^s(0,t)$ is given by Corollary 3.2.

Therefore,

$$P(\tau^k \le t|\mathcal{H}_t) = \sum_{i=0}^{m-k}\sum_{h=i}^{m} \binom{h}{i} (-1)^{h-i} \sum_{s\in\mathcal{S},|s|=h} e^{-\sum\limits_{j\in s}\Lambda_t^j}, \tag{4.4}$$

and,

$$P(\tau^k \le t) = \sum_{i=0}^{m-k}\sum_{h=i}^{m} \binom{h}{i} (-1)^{h-i} \sum_{s\in\mathcal{S},|s|=h} \langle \Psi_3^s(0,t)\mathbf{1}, X_0\rangle.$$

Similarly, the conditional survival distribution of τ^1 is,

$$P(\tau^1 > t|\mathcal{H}_t) = P(\tau_1 > t, \cdots, \tau_m > t|\mathcal{H}_t) = e^{-\sum\limits_{l=1}^{m}\Lambda_t^l}.$$

Furthermore, from Corollary 3.2, it follows that,

$$P(\tau^1 > t) = P(\tau_1 > t, \cdots, \tau_m > t) = \langle \Psi_4(0,t)\mathbf{1}, X_0\rangle,$$

where $\Psi_4(0,t)$ is defined in Corollary 3.2.

Next, using the conditional distribution of τ^k, we will calculate a kth-to-default CDS spread s^k. The cash flows of a kth-to-default CDS are as follows:

Default leg: the protection seller covers the credit losses $1-R$ as soon as k assets of the underlying portfolio have defaulted;

Premium leg: the protection buyer pays $s^k\Delta t_i$ to the seller, at each date $t_i, i = 1, \cdots, M$ until maturity or until k assets of the underlying portfolio default before maturity.

Then, the kth-to-default CDS spread s^k is determined so that the discounted payoff of the two legs are equal when the contract is settled at the initial time. More precisely, the kth-to-default CDS spread s^k should satisfy

$$s^k E[\sum_{l=1}^{M} e^{-\int_0^{t_l} r_u du}\triangle t_l 1_{\{\tau^k > t_l\}}] = (1-R)E[\sum_{l=1}^{M} e^{-\int_0^{t_l} r_u du} 1_{\{t_{l-1} < \tau^k \le t_l\}}].$$

Hence,

$$s^k = \frac{(1-R)E[\sum\limits_{l=1}^{M} e^{-\int_0^{t_l} r_u du} 1_{\{t_{l-1} < \tau^k \le t_l\}}]}{E[\sum\limits_{l=1}^{M} e^{-\int_0^{t_l} r_u du}\triangle t_l 1_{\{\tau^k > t_l\}}]}. \tag{4.5}$$

Proposition 4.2 *The spread of the kth-to-default CDS, s^k, is given by*

$$s^k = \frac{\sum_{l=1}^{M}\sum_{i=m-k+1}^{m}\sum_{h=i}^{m}\left(\begin{array}{c} h \\ i \end{array}\right)(-1)^{h-i}\sum_{s\in\mathcal{S},|s|=h}(P_{1l}^{hs}-P_{2l}^{hs})}{\sum_{l=1}^{M}\triangle t_l\sum_{i=m-k+1}^{m}\sum_{h=i}^{m}\left(\begin{array}{c} h \\ i \end{array}\right)(-1)^{h-i}\sum_{s\in\mathcal{S},|s|=h}P_{2l}^{hs}}, \tag{4.6}$$

where

$$P_{1l}^{hs} = \langle\tilde{\Psi}^{hs}(0,t_{l-1})(e^{(Q-\mathbf{diag(r)})\Delta t_l}\mathbf{1}), X_0\rangle,$$

and

$$P_{2l}^{hs} = \langle\tilde{\Psi}^{hs}(0,t_l)\mathbf{1}, X_0\rangle.$$

Here $\tilde{\Psi}^{hs}(u,t)$ is the fundamental matrix solution of

$$\frac{d\tilde{\Psi}^{hs}(u,t)}{du} + \tilde{O}^{hs}(u)\tilde{\Psi}^{hs}(u,t) = 0, \tilde{\Psi}^{hs}(t,t) = \mathbf{I}. \tag{4.7}$$

where $\tilde{O}^{hs}(u) = (\tilde{o}_{ij}^{hs}(u))$ is an $N\times N$ matrix:

$$\begin{aligned}\tilde{o}_{ij}^{hs}(u) &= q_{ij}e^{-\sum_{|s|=h,l\in s} w_{ij}^l}, i\neq j,\\ &= q_{ii} + \overline{\hat{G}}_u^{hsi} - r_i - \sum_{|s|=h,l\in s}\lambda^{li}, i = j,\end{aligned}$$

with

$$\overline{\hat{G}}_u^{hsi} = \sum_{l\in s,|s|=h}\mu_l^i(\int_0^\infty e^{\overline{B}^l(u,t)x}f^{li}(x)dx - 1) + \mu_0^i(\prod_{l\in s,|s|=h}\int_0^\infty e^{\overline{B}^l(u,t)x}f^{li}(x)dx - 1),$$

and $\overline{B}^l(u,t)$ given in Corollary 3.1.

Proof. The proof is similar to the one of Proposition 4.1. We first calculate the discounted payoff. Using the "tower property" of conditional expectation and (4.4), we have

$$\begin{aligned}&E[e^{-\int_0^{t_l} r_u du}1_{\{t_{l-1}<\tau^k\le t_l\}}] = E[e^{-\int_0^{t_l} r_u du}P(t_{l-1}<\tau^k\le t_l|\mathcal{H}_{t_l})]\\ \hat{=}\ &\sum_{i=m-k+1}^{m}\sum_{h=i}^{m}\left(\begin{array}{c} h \\ i \end{array}\right)(-1)^{h-i}\sum_{s\in\mathcal{S},|s|=h}(P_{1l}^{hs}-P_{2l}^{hs}),\end{aligned}$$

where

$$P_{1l}^{hs} = E[e^{-\int_0^{t_l} r_u du - \sum_{i\in s,|s|=h}\Lambda_{t_{l-1}}^i}]$$

and

$$P_{2l}^{hs} = E[e^{-\int_0^{t_l} r_u du - \sum_{i\in s,|s|=h}\Lambda_{t_l}^i}].$$

Therefore, it remains to calculate P_{1l}^{hs} and P_{2l}^{hs}. Again using the "tower property" of conditional expectation yields

$$\begin{aligned}P_{1l}^{hs} &= E\big[e^{-\int_0^{t_{l-1}} r_u du - \sum_{i\in s,|s|=h}\Lambda_{t_{l-1}}^i}E[e^{-\int_{t_{l-1}}^{t_l} r_v dv}|\mathcal{H}_{t_l}^X\vee\mathcal{H}_{t_{l-1}}]\big]\\ &= \langle e^{(Q-\mathbf{diag(r)})\Delta t_l}\mathbf{1}, E[e^{-\int_0^{t_{l-1}} r_u du - \sum_{i\in s,|s|=h}\Lambda_{t_{l-1}}^i}X_{t_{l-1}}]\rangle\\ &= \langle\tilde{\Psi}^{hs}(0,t_{l-1})(e^{(Q-\mathbf{diag(r)})\Delta t_l}\mathbf{1}), X_0\rangle,\end{aligned}$$

where the second equality follows from (3.9), and the third equality is a direct consequence of Theorem 3.1.

The expression for P_{2l}^{hs} can be directly obtained from Theorem 3.1,

$$P_{2l}^{hs} = \langle \tilde{\Psi}^{hs}(0, t_l)\mathbf{1}, X_0 \rangle.$$

The discounted payoff of the premium leg is

$$\begin{aligned} E[e^{-\int_0^{t_l} r_v du} 1_{\{\tau_k > t_l\}}] &= E[e^{-\int_0^{t_l} r_v dv} P(\tau_k > t_l | \mathcal{H}_{t_l})] \\ &= \sum_{i=m-k+1}^{m} \sum_{h=i}^{m} \begin{pmatrix} h \\ i \end{pmatrix} (-1)^{h-i} \sum_{s \in \mathcal{S}, |s|=h} P_{2l}^{hs}. \end{aligned}$$

Then substituting the expressions for P_{1l}^{hs} and P_{2l}^{hs} into the discounted payoffs of the premium leg and the default leg, we end the proof. □

Then from Proposition 4.2, we can directly obtain the following result.

Corollary 4.1 *The spread of the first-to-default CDS, s^1, is given by*

$$s^1 = \frac{\sum_{l=1}^{M} (P_{1l} - P_{2l})}{\sum_{l=1}^{M} \triangle t_l P_{2l}}, \tag{4.8}$$

where

$$P_{1l} = \langle \Phi(0, t_{l-1})(e^{(Q - \mathbf{diag}(\mathbf{r}))\Delta t_l} \mathbf{1}), X_0 \rangle,$$

and

$$P_{2l} = \langle \Phi(0, t_l)\mathbf{1}, X_0 \rangle.$$

Here $\Phi(u, t)$ is the fundamental matrix solution of

$$\frac{d\Phi(u, t)}{du} + \hat{O}(u)\Phi(u, t) = 0, \Phi(t, t) = \mathbf{I}. \tag{4.9}$$

where $\hat{O}(u) = (\hat{o}_{ij}(u))$ is an $N \times N$ matrix:

$$\begin{aligned} \hat{o}_{ij}(u) &= q_{ij} e^{-\sum_{l=1}^{m} w_{ij}^l}, i \neq j, \\ &= q_{ii} + \hat{G}_u^i - r_i - \sum_{l=1}^{m} \lambda^{li}, i = j, \end{aligned}$$

with $\hat{G}_u^i$ defined in (3.15).

□

5. Numerical results

In this section, we carry out a numerical study to examine the impact of some model parameters on the spreads of the CDS.

For the simplicity of computation, we consider a single-name CDS and the first-to-default spread of a homogenous portfolio of $m = 10$ credit-risky assets. We set $\triangle t_l = \frac{1}{4}$. Following Giesecke et al.

(2011), we consider $N = 3$; that is, X has three states, where state e_1, state e_2, and state e_3 represent a "good" economy, a "moderate" economy, and a "bad" economy, respectively. Also, the generator of the Markov chain can be borrowed from Giesecke et al. (2011). We set

$$Q = \begin{pmatrix} -0.10474 & 0.08865 & 0.01609 \\ 0.84799 & -0.848 & 0.00001 \\ 0.69561 & 0.00001 & -0.69562 \end{pmatrix}$$

As is reported in Giesecke et al. (2011), we set $\boldsymbol{\lambda}^1 = (0.00741, 0.004261, 0.11137)^*$. Generally speaking, the main features of the financial market in a "bad ("good) economy are high (low) default probability and low (high) interest rate. Hence, we set $\mathbf{r} = (0.05, 0.03, 0.01)^*$, $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_0 = (1, 3, 5)^*$, and f_t^1 is given by

$$f^{11}(x) = 2000e^{-2000x}, x > 0; f^{12}(x) = 1000e^{-1000x}, x > 0; f^{13}(x) = 500e^{-500x}, x > 0.$$

Let $w_{12} = -w_{21} = w_{23} = -w_{32} = w, w_{13} = -w_{31} = 2w$. In order to investigate the impact of the jumps of the Markov chain on the spread, we shall study the relationship between the spread and the parameter w. We also study how the exponential decay rate a impacts the spread.

Figure 1 plots the term structure of the spread s_1 for $w = 0.01$ and $w = 0$, respectively. We can see from it that the spread s_1 for $w = 0.01$ is larger than the one for $w = 0$ when we start at a "good" economy. This is because when a "good" economy switches to a "moderate" or a "bad" economy, the hazard process corresponding to $w = 0.01$ will increase by a positive amount, which leads to a larger default probability and therefore a larger spread. On the contrary, if we start at a "moderate" or a "bad" economy, the spread s_1 for $w = 0.01$ is smaller than the one for $w = 0$, since a transition from a "moderate" or a "bad" economy to a "good" economy will cause the hazard process corresponding to case of $w = 0.01$ to jump downward by a positive amount, which implies a smaller default probability and therefore a lower spread. We can also see that the difference between the spreads is very small when we start at a "good" economy, while it is very large when we start at a "moderate" or a "bad" economy. As is pointed out by Giesecke et al. (2011), the regime e_1 is very persistent and the other two regimes are much less persistent. Therefore, if we start at e_1, then the probability of migrating to e_2 or e_3 is very small, while if we start at e_2 or e_3, then the probability of migrating to e_1 is very large.

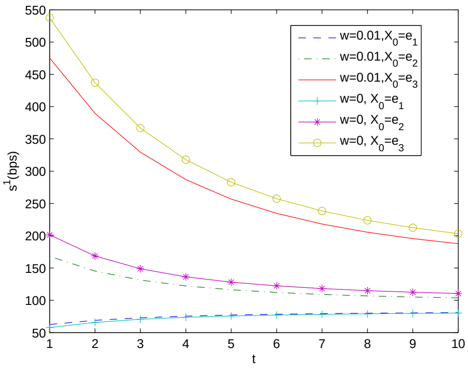

Figure 1: Relationship between s_1 and t, for $a = 1$

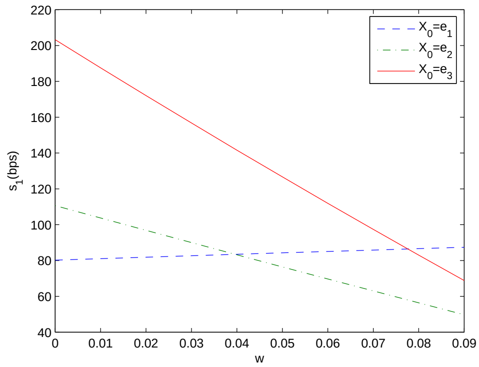

Figure 2: Relationship between s_1 and w with $a = 1, T = 10$.

Figure 2 presents the relationship between s_1 and w for a fixed maturity $T = 10$. From it we can see the spread s_1 decreases with w when we start from $X_0 = e_2$ or $X_0 = e_3$. This is because from the generator of the Markov chain, we can conclude that if we start from $X_0 = e_2, (X_0 = e_3)$ then the probability of switching to e_1 is much larger than the probability of switching to $e_3(e_2)$ or staying

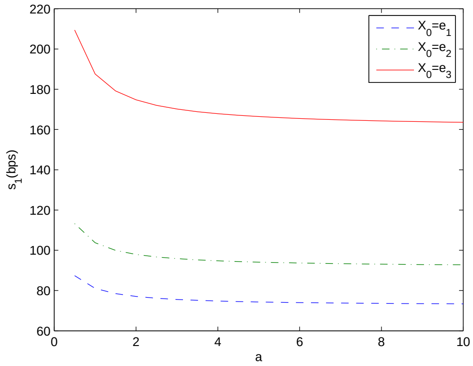

Figure 3: Relationship between s_1 and a with $w = 0.01, T = 10$.

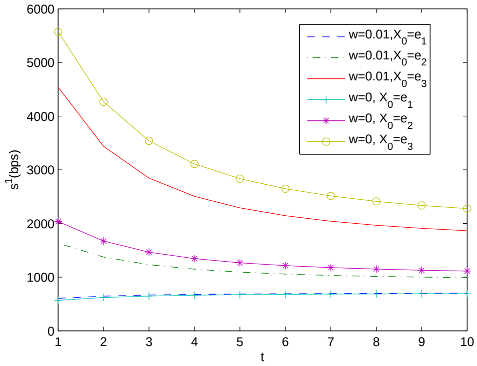

Figure 4: Relationship between s^1 and w with $q = 0.5, T = 5$.

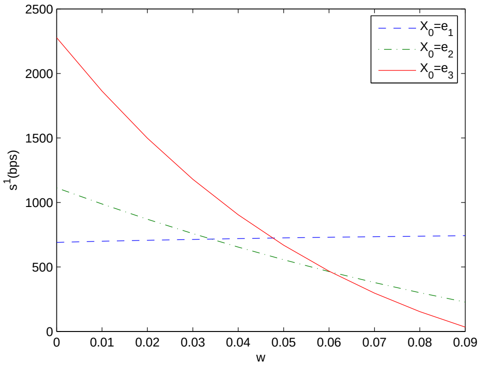

Figure 5: Relationship between s^1 and w with $q = 0.5, T = 5$.

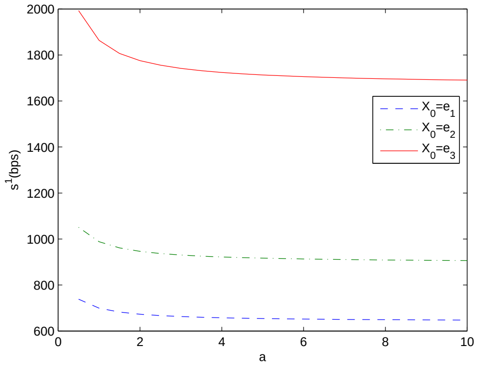

Figure 6: Relationship between s^1 and w with $q = 0.5, T = 5$.

$e_2(e_3)$. Therefore, a larger value of w will cause an increasing number of downward jumps in the hazard process, and therefore a smaller value of the hazard process. We can also see that although s_1 increases with w when we start from $X_0 = e_1$, it does not change much. This is because the regime e_1 is very persistent and then the probability of switching to the other two regimes is very small.

Figure 3 illustrates the relationship between the spread s_1 and a for a fixed maturity $T = 10$. We can observe that s_1 decreases with a. This is because a larger decay rate a implies a shorter time period needed to go back to the previous level of λ_t^i immediately after shock events occur. That is to say the value of λ_t^i decreases with a, which leads to a smaller value of hazard process and a smaller spread.

Figure 4 plots the term structure of the spread s^1 for $w = 0.01$ and $w = 0$, respectively. Figures 5-6 present the impacts of the parameters w and a on the first-to-default basket swap spread. From them, we can see that the curves are similar to those of Figures 1-3, and the first-to-default basket swap spread is much higher than the spread for a single CDS s_1, which is consistent with the the stylized features and the financial intuition. The above numerical results illustrate that if we ignore the jumps driven by the transition between regimes in the hazard process, we may underprice the credit derivatives in a “good” economy and overprice them in a “moderate” or a “bad” economy. Especially, when we start at a “moderate” or a “bad” economy, the prices of the credit derivatives will be seriously overpriced.

6. Concluding remarks

In this paper, we use a conditionally independent approach to analyze a single-name CDS contract and a kth-to-default basket swap. Our study contributes to the credit risk literature by by providing a correlated default model that incorporates both macroeconomic risks and firm-specific jump risks. The hazard processes are modelled by some regime-switching pure jump processes, in which not only the model parameters may switch but also the hazard processes may jump whenever transitions in the Markov chain occur. A key feature of our regime-switching model is that the defaults can be triggered by the jumps of the Markov chain. Furthermore, if the Markov chain jumps from a state of economic growth to a state of recession, this may lead the conditional default intensity of all the firms to go up, increasing the chances of observing a larger number of defaults in the portfolio. Therefore, the proposed hazard processes can well capture the clustering phenomena in correlated defaults.

The default dependence structure we construct stems from three sources. First, the hazard processes of the two firms are both affected by a Markov chain, which describes the impact of the market regimes on the the default probability. Second, default dependence arises from common jumps in the processes λ_t^i modelled by a regime-switching compound Poisson process, which models the impact of some common factors other than market regimes on the default probability. Finally, dependent structure arises from conditional independence. We give the joint Laplace transform of the hazard processes via a martingale method. Based on the Laplace transform, we can calculate the CDS spread and the kth-to-default basket swap spread. Numerical results illustrate that ignoring the jumps driven by the regime switches will lead to underpricing or overpricing the credit derivatives.

Since the model is still numerically tractable and the number of parameters of the model is flexible, it will be suitable for many applications in the field of risk management and actuarial applications. See for example, we can consider a dependent mortality structure by using a similar regime-switching model. An interesting open problem is the estimation of the model from market data.

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